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Generalized formulas for ray-tracing and longitudinal spherical aberration

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In this work, we generalize the paraxial ray-tracing formulas to include nonparaxial rays. For a refracting (reflecting) spherical surface, a new single meridional formula is derived. This formula can be easily reduced to a paraxial formula. It can also be applied to any aspheric (or general) surface with a known equation. Also, a new exact ray-tracing procedure for a centered system of spherical surfaces is derived. In this procedure, we apply just two simple equations for each surface of the system, which, to the best of our knowledge, makes it the shortest analytical ray-tracing technique ever. This procedure can be applied in some other applications. For example, it can be reduced to a new single paraxial formula that can be easily used to trace a paraxial ray propagating through a system of spherical surfaces. Also, it is applied to derive an exact meridional formula for both thick and thin lenses that can also be reduced to a new paraxial formula different from the Gaussian one. These results led us to easily derive an exact formula for the longitudinal spherical aberration for both thick and thin lenses and also for a single refracting (reflecting) spherical surface. Numerical examples are provided and discussed.

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1. INTRODUCTION

In the literature of geometrical optics, there are several exact ray-tracing procedures suggested [1–7]. In most cases, any one of these procedures is presented as a series of equations that is applied to each surface of an optical system. For many decades, these procedures were continuously developed and widely used in the field of optical design. At the present time, there are several high-quality, ray-tracing computer programs available to the optical designer. The ultimate goal of optical design is to reduce ray aberrations to get the sharpest image possible [2,3]. However, practical applications should not be the only task of exact ray tracing. Another possible task is to help us understand the behavior of light rays in the nonparaxial region. In Gaussian optics, we have a group of simple linear equations describing the behavior of paraxial rays in optical systems, but we really miss a similar group of exact equations for nonparaxial rays. All optical engineers and optics students know these paraxial formulas and use them frequently, but they never see their generalized forms. Each of these generalized forms should be reducible to the paraxial form in a few simple steps. Mathematically, it also should be containable in the body of any theoretical work concerning optical design or any other topics. The present work aims to derive some of these exact nonparaxial formulas using simple mathematics. We think that deriving such formulas can be achieved successfully if we avoid using the concept of principal planes in the paraxial region. We replace the principal planes by the curvature centers of the optical surfaces. For example, in the case of refraction at a spherical surface, we determine the positions of the object and the image by measuring the distances between each of them and the curvature center of the surface. For the case of a thick lens, the locations of the object and the image are determined by measuring the distances between each of them and the curvature centers of the first and second surfaces of the lens, respectively, and not to the principal planes. This way, the obtained meridional formulas can be easily reduced to simple paraxial forms. In the present work, our attention is mainly focused on meridional rays and spherical surfaces. The aim of the work is not to design a ray-tracing procedure for computer software and commercial purposes, but to explore how such general formulas can be derived, how they will seem, and how they may help as an introduction to a theory for nonparaxial optics.

2. SIGN CONVENTION

(1) Light rays travel from left to right.
(2) The radius of curvature, R, is positive for a convergent (refracting/reflecting) surface and vice versa.
(3) The distance S_o between the object and the vertex V of the surface is positive when the object is located to the left of V and vice versa.

(4) The distance S_1 between the image I and the vertex V of the surface is positive when the image is located to the right (left) of the refracting (reflecting) surface and vice versa.

(5) The distance K_o between the object and the center of curvature C for a (reflecting/refracting) surface is positive if the object is to the left of C and vice versa.

(6) The distance K_1 between the image and the center of curvature C for a (reflecting/refracting) surface is positive if the image is to the right of C and vice versa.

(7) The angle that the ray makes with the optical axis is positive if the latter has to be rotated in the counterclockwise direction (through the acute angle) to coincide with the ray and vice versa.

3. REFRACTION AT A REFRACTING SURFACE

A. Description

Figure 1 shows a spherical surface separating two media of refractive indices n_o and n_1. Let us assume that O is a point object and I is its image formed by the surface. Let K_o and K_1 be the distances of both O and I measured from the center of curvature C. In the present work, all distances are measured to the centers of curvature of the refracting (reflecting) surfaces. This greatly simplifies the mathematical work. The incident ray OP makes an angle θ_o with the optical axis, while the refracted ray PI makes an angle θ_1.

Our target now is to calculate the location of the image I (i.e., to calculate the distance K_1). In geometrical optics, we are used to solving this problem by using a series of trigonometric equations. In this work, we will prove that the distance K_1 can be calculated using the formula

$$\frac{1}{K_1} = \frac{n}{aR} \sin\left[\sin^{-1} a - \sin^{-1}\frac{a}{n} - \theta_o\right], \quad (1)$$

where

$$a = \frac{K_o \sin\theta_o}{R}. \quad (2)$$

Equation (1) is a new exact meridional formula for refraction at a spherical surface. It can be easily used to calculate the image distance K_1 by direct substitution, as will be shown in a numerical example. The derivation of Eq. (1) is based on the geometry of Fig. 1. Let φ and θ be the angles of incidence and refraction, respectively. According to the sign convention mentioned above, the angle θ_o is positive, while θ_1 is negative. So, we can write

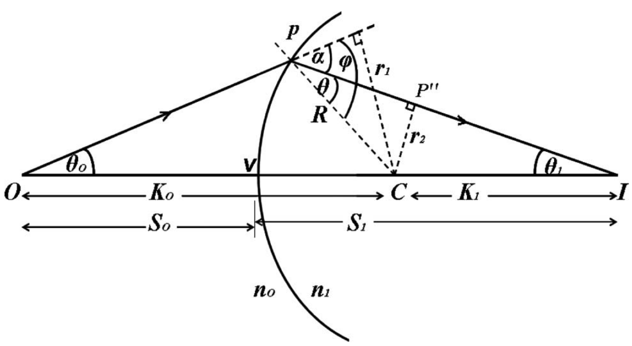

Fig. 1. Meridional refraction at a spherical surface.

$$\alpha = \theta_o - \theta_1 = \varphi - \theta, \quad (3)$$

where α is the deviation angle. Therefore,

$$\theta_1 = \theta_o - \varphi + \theta. \quad (4)$$

For simplicity, we can take $n = n_1/n_o$, where n is the relative refractive index and $n_1 > n_o$. So, we can write $\sin\varphi = n \sin\theta$. Now let

$$a = \sin\varphi. \quad (5)$$

Then, $\varphi = \sin^{-1} a$ and $\theta = \sin^{-1} a/n$. Thus Eq. (4) can be written as

$$\theta_1 = \theta_o - \sin^{-1} a + \sin^{-1} a/n. \quad (6)$$

In Fig. 1, we have

$$\sin\varphi = \frac{r_1}{R} = \frac{K_o \sin\theta_o}{R}. \quad (7)$$

Therefore,

$$a = \frac{K_o \sin\theta_o}{R}. \quad (8)$$

And from the triangle $CP''I$ we have

$$\frac{K_1}{\sin 90} = \frac{r_2}{\sin(-\theta_1)}. \quad (9)$$

The relation between r_1 and r_2 can be easily found as follows: $\sin\varphi = r_1/R$ and $\sin\theta = r_2/R$. In this case, $\sin\varphi/\sin\theta = r_1/r_2$ and by Snell's law, this directly leads to $r_1 = nr_2$. Therefore,

$$r_2 = \frac{K_o \sin\theta_o}{n}. \quad (10)$$

From Eq. (9), we get

$$\frac{1}{K_1} = \frac{n}{K_o \sin\theta_o} \sin(-\theta_1). \quad (11)$$

Finally, we substitute from Eqs. (6) and (8) into Eq. (11) to get Eq. (1).

B. Numerical Examples

(1) In Fig. 1, let us calculate the distance K_1 of the image I from the center of curvature of a refractive spherical surface if $K_o = 18$ cm, $R = 8$ cm, $n = 2$ and $\theta_o = 15.825489$. Using Eq. (2), we get $a = 0.613593627$. Then we substitute into Eq. (1) to easily get $K_1 = 33.849960$ cm.

(2) For the case of a concave (divergent) surface, let us consider the following numerical example: $R = -8$ cm, $K_o = 12$ cm, $n = 2$, and $\theta_o = 8.783323$. Substituting in Eqs. (1) and (2) we get $a = -0.229047$ and $K_1 = -3.439578$ cm. According to the sign convention, K_1 must be negative, since the image is formed to the left of the curvature center. To calculate the distance S_1 of the image I from the vertex v of the surface, we just add R to K_1 (i.e., $S_1 = K_1 + R = -3.439578 - 8 = -11.439578$ cm).

Remarks:

(1) Equation (1) is not only used to determine the position of the image, but it can also be reduced to the paraxial formula, as we shall see in the next section. It also can be used to get an exact spherical mirror meridional formula.

(2) This is right for both meridional and skew rays. For example, if the point O in Fig. 1 is located out of the optical axis and Op is a skew ray, then we can consider a secondary optical axis OC passing through any point C located at the normal to the surface. In this case, the plane OPC is considered as a secondary meridional plane and Eq. (1) can be applied to get K_2 of the image that will be formed at the secondary optical axis. The details of this idea will be published in a future work.

C. Paraxial Approximation

In the paraxial region, the ray's inclination angles are very small. So, for any angle x we have $\sin x = x$ and $\sin^{-1} x = x$. Therefore, Eq. (2) reduces to

$$a = \frac{K_o \theta_o}{R}. \tag{12}$$

In this case, $a \simeq 0$. So, substituting into Eq. (1) we get

$$\frac{1}{K_1} = \frac{n}{aR}\left[a - \frac{a}{n} - \theta_o\right], \tag{13}$$

$$\frac{1}{K_1} + \frac{n}{K_o} = \frac{n-1}{R}. \tag{14}$$

This is the paraxial formula written in terms of the distances K_o and K_1. It also can be written in the form

$$\frac{1}{S_1 - R} + \frac{n}{S_o + R} = \frac{n-1}{R}, \tag{15}$$

where the distances S_o and S_1 are shown in Fig. 1. The two paraxial formulas (14) and (15) are equivalent to the Gaussian one,

$$\frac{1}{S_o} + \frac{n}{S_1} = \frac{n-1}{R}. \tag{16}$$

So, we have three formulas for the single spherical surface.

D. Parallel Incidence

When the incident rays are parallel to the optical axis, then

$$r_1 = K_o \sin\theta_o = h, \tag{17}$$

where h is the height of the point of incidence P from the optical axis. In this case, we find that $a = h/R$ and Eq. (1) will have the form

$$\frac{1}{K_1} = \frac{n}{h}\sin\left[\sin^{-1}\frac{h}{R} - \sin^{-1}\frac{h}{nR}\right]. \tag{18}$$

This is the parallel incidence equation in the trigonometric form. The nontrigonometric form can be obtained by expanding the sin function in Eq. (18) and this leads to the formula

$$K_1 = \frac{\pm R^2}{\sqrt{n^2R^2 - h^2} - \sqrt{R^2 - h^2}}. \tag{19}$$

In Eq. (19), the positive sign is chosen for the convergent refractive surface and vice versa.

4. REFLECTION AT A SPHERICAL MIRROR

A. Description

In Fig. 2, O is a point object located at a distance K_o from the curvature center C of a spherical mirror, and I is the image

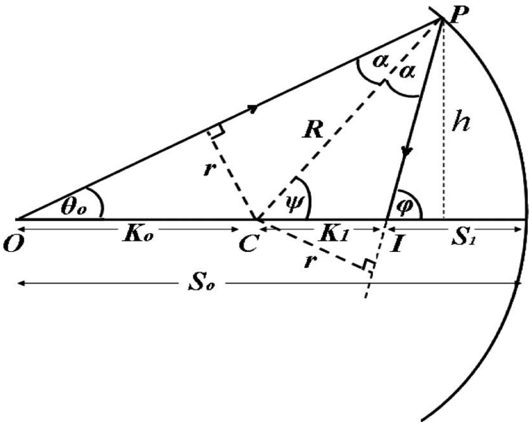

Fig. 2. Meridional reflection at a spherical mirror.

formed by the mirror for this object at a distance K_1 from C. Since

$$\frac{1}{K_1} = \frac{1}{r}\sin\phi, \tag{20}$$

where r is given by $r = K_o \sin\theta_o$, therefore,

$$\frac{1}{K_1} = \frac{1}{K_o \sin\theta_o}\sin[\theta_o + 2\alpha]. \tag{21}$$

Now,

$$\sin\alpha = (K_o \sin\theta_o)/R. \tag{22}$$

Thus, Eq. (21) can be written as

$$\frac{1}{K_1} = \frac{1}{K_o \sin\theta_o}\sin\left[\theta_o + 2\sin^{-1}\frac{K_o \sin\theta_o}{R}\right]. \tag{23}$$

This is the meridional formula of a spherical mirror for any angle θ_o. It can be used to derive another nontrigonometric formula for the mirror having the form

$$-\frac{1}{K_o} + \frac{1}{K_1} = \frac{2}{R}\sqrt{1 - \left(\frac{h}{R}\right)^2}, \tag{24}$$

where h is the height of the point of incidence from the optical surface. To derive Eq. (24), we use (21) and (22) to get

$$\left.\begin{aligned}\tfrac{1}{K_1} &= \tfrac{1}{R\sin\alpha}\sin[\theta_o + 2\alpha]\\ &= \tfrac{1}{R\sin\alpha}[\sin\theta_o\cos(2\alpha) + \cos\theta_o\sin(2\alpha)]\end{aligned}\right\}. \tag{25}$$

Using the identities

$$\begin{aligned}\cos(2\alpha) &= 1 - 2\sin^2\alpha\\ \sin(2\alpha) &= 2\sin\alpha\cos\alpha,\end{aligned} \tag{26}$$

we get

$$\begin{aligned}\frac{1}{K_1} = \frac{1}{R\sin\alpha}[&\sin\theta_o - 2\sin\theta_o\sin^2\alpha\\ &+ 2\cos\theta_o\sin\alpha\cos\alpha],\end{aligned} \tag{27}$$

$$\frac{1}{K_1} = \frac{\sin\theta_o}{R\sin\alpha} - \frac{2\sin\alpha}{R\sin\alpha}[\sin\theta_o\sin\alpha - \cos\theta_o\cos\alpha]. \tag{28}$$

Using Eq. (21) we get

$$\frac{1}{K_1} = \frac{1}{K_o} + \frac{2}{R}\cos[\theta_o + \alpha]. \quad (29)$$

In Fig. 2, we have

$$\cos[\theta_o + \alpha] = \cos\psi = \sqrt{R^2 - h^2}/R. \quad (30)$$

By substituting into Eq. (29), we finally get Eq. (24), which is the simplest meridional equation for a spherical mirror.

B. Another Derivation to the Mirror Equation

Equation (24) can be derived following simpler steps. In Fig. 2, we see that $\varphi = \alpha + \psi$, and $\theta_o = \psi - \alpha$. So, we have

$$\sin\varphi = \sin\alpha\cos\psi + \cos\alpha\sin\psi, \quad (31)$$

$$\sin\theta_o = \sin\psi\cos\alpha - \cos\psi\sin\alpha. \quad (32)$$

Subtracting Eq. (32) from Eq. (31), we get

$$\sin\varphi - \sin\theta_o = 2\sin\alpha\cos\psi. \quad (33)$$

It is clear that $\sin\varphi = r/K_1$, $\sin\theta_o = r/K_o$, $\sin\alpha = r/R$, and $\cos\psi = \sqrt{1 - h^2/R^2}$. So, substituting into (33), we get Eq. (24).

Remarks:

(1) It should be noted that Eq. (23) can be, directly, obtained from Eq. (1). In this case, we must remember that Eq. (1) was derived for a convex refracting surface. So, according to the sign convention, R is negative for a convex mirror, and $n = -1$. Thus, Eq. (1) reduces directly to Eq. (23).

(2) Equations (1), (23), and (24) can be applied also for skew rays. For example, if the point O in Fig. 2 is located out of the optical axis and Op is a skew ray, then we can consider a secondary optical axis OC passing through any point C located at the normal to the surface. In this case, the plane OPC is considered as a secondary meridional plane and Eqs. (1), (23), or (24) can be applied to get K_2 of the image formed at the secondary optical axis OC. An extension of this work to the case of skew rays is to appear in the near future.

C. Numerical Examples

In Fig. 2, let us calculate K_1 for the image formed by the concave mirror if we know that $R = +6$ cm, $K_o = 7.2$ cm, $\theta_o = 22.5^\circ$. This problem can be easily solved by direct substitution in Eq. (23), where we get $K_1 = 2.825833$ cm. If the height h is given instead of θ_o, substitution into Eq. (24) leads to the same result.

D. Paraxial Approximation

In the paraxial region, we consider that $\theta_o \simeq 0$ in Eq. (23). So, $\sin\theta_o \simeq \theta_o$, and the quantity between brackets also tends to zero. Thus, Eq. (23) reduces to

$$\frac{1}{K_1} = \frac{1}{K_o\theta_o}\left[\theta_o + \frac{2K_o\theta_o}{R}\right]. \quad (34)$$

So, we get

$$\frac{-1}{K_o} + \frac{1}{K_1} = \frac{2}{R}. \quad (35)$$

Equation (35) is the paraxial formula of reflection at a spherical mirror. It can be also written in the form

$$\frac{1}{R - S_o} + \frac{1}{R - S_1} = \frac{2}{R}, \quad (36)$$

where S_o and S_1 are the distances of both the object O and the image I from the mirror. So, Eqs. (35) and (36) are equivalent to the famous Gaussian formula,

$$\frac{1}{S_o} + \frac{1}{S_1} = \frac{2}{R}. \quad (37)$$

So, we have three forms for the paraxial equation of a spherical mirror. It should be noted that Eq. (24) of the mirror can also be reduced to the paraxial form by just putting $h = 0$.

5. REFRACTION (REFLECTION) AT ASPHERIC SURFACES

Equations (1) and (24) can be applied for the case of aspheric refracting (reflecting) surfaces. For example, in Fig. 3, a meridional ray is refracted at a surface with equation $f(x, y) = 0$ (in two dimensions). If the height h of the point of incidence is known, we can use Eq. (1) to calculate K_1 as follows: The three quantities K_o, θ_o, and A can be easily calculated as follows: solving $f(x_o, h) = 0$ for x_o, then we can get $\theta_o = \cot^{-1}(x_o/h)$. The angle γ is given by $\gamma = \tan^{-1} 1/f'(x_o, h)$. The length A, which corresponds to R in Fig. 1, is normal to the tangent at the point of incidence. So, we have $A = h/\sin\gamma$ and $x_1 = h\cot\gamma$. Therefore, the distance K_o is given by $K_o = x_o + x_1$. Now, substituting in Eq. (1) for K_o, θ_o, and A we get K_1. For the case of reflection, we follow similar steps using Eqs. (23) and (24).

6. RAY-TRACING PROCEDURE FOR A CENTERED SYSTEM OF SPHERICAL SURFACES

A. Description

In the literature, there are several procedures for meridional ray tracing. In an exact ray-tracing procedure, we usually use a series of trigonometric (or vector) equations for each one of the successive optical surfaces. In this section, we describe a new simple procedure in which we use just two equations for each one of the surfaces. It should be stressed here that the technique only works in the case of spherical surfaces. We believe this procedure will be the shortest and fastest analytical technique published in the literature.

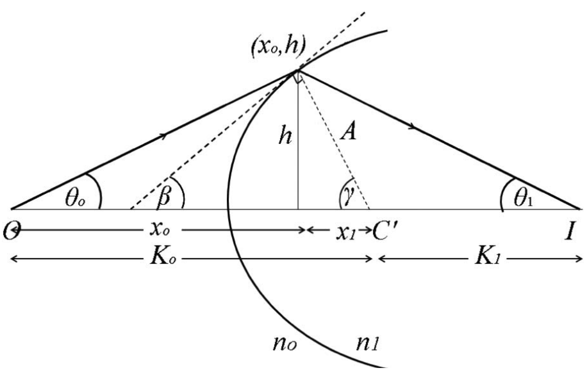

Fig. 3. Meridional ray refraction at aspheric surface with equation $f(x, y) = 0$.

B. Procedure Derivation

In Fig. 4, we have two spherical refracting surfaces. The object O is located at a distance K_o from the center of curvature of the first surface. Assuming that φ_1 and φ_1' (not shown in figure) are the angles of incidence and refraction, respectively, the deviation angle of the first surface is

$$\delta_1 = \varphi_1 - \varphi_1'. \tag{38}$$

Now, let θ_o and θ_1 be the angles made by the incident and refracted rays, respectively, with the optical axis. In this case, we have

$$\delta_1 = \theta_o - \theta_1. \tag{39}$$

Thus,

$$\theta_1 = \theta_o - \varphi_1 + \varphi_1'. \tag{40}$$

Let $a_1 = \sin\varphi_1$; thus, $\varphi_1 = \sin^{-1} a_1$ and $\varphi_1' = \sin^{-1}\frac{n_o a_1}{n_1}$.

Also, we have to note that $a_1 = \sin\varphi_1 = \frac{K_o \sin\theta_o}{R_1}$. Thus, for the first optical surface, we get the following two equations:

$$a_1 = \frac{K_o \sin\theta_o}{R_1}, \tag{41}$$

$$\theta_1 = \theta_o - \sin^{-1} a_1 + \sin^{-1}\frac{n_o a_1}{n_1}. \tag{42}$$

Equations (41) and (42) are sufficient for the first surface. We do not need to calculate the image position formed by this surface, because this will be done only for the last surface of the optical system. For the second surface (see Fig. 4), similar steps are followed. We should remember that θ_2 is negative according to the sign convention given in Section 1. Now, the deviation angle δ_2 is given by

$$\delta_2 = \theta_1 - \theta_2 = \varphi_2 - \varphi_2', \tag{43}$$

where φ_2 and φ_2' are the angles of incidence and refraction at the second surface, respectively. So, we can write

$$\theta_2 = \theta_1 - \varphi_2 + \varphi_2'. \tag{44}$$

Let $a_2 = \sin\varphi_2$, so we have $\varphi_2 = \sin^{-1} a_2$ and $\varphi_2' = \sin^{-1}\frac{n_1 a_2}{n_2}$, and Eq. (44) can be written as

$$\theta_2 = \theta_1 - \sin^{-1} a_2 + \sin^{-1}\frac{n_1 a_2}{n_2}. \tag{45}$$

Figure 4 shows that

$$a_2 = \sin\varphi_2 = \frac{r_3}{R_2} = \frac{r_2 + d_1 \sin\theta_1}{R_2}, \tag{46}$$

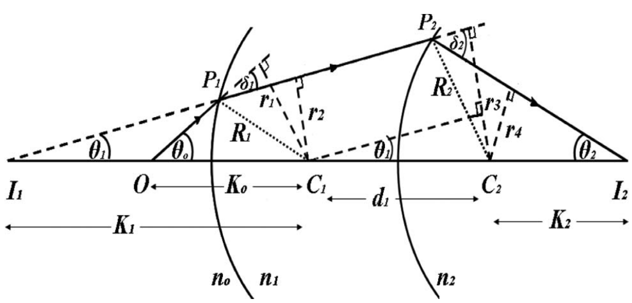

Fig. 4. Meridional refraction through a system of two spherical surfaces.

$$a_2 = \frac{(n_o/n_1) r_1 + d_1 \sin\theta_1}{R_2} = \frac{n_o r_1}{n_1 R_2} + \frac{d_1 \sin\theta_1}{R_2}, \tag{47}$$

where $r_2 = (n_o/n_1) r_1$, as was proved in Section 3.A, and $r_1 = K_o \sin\theta_o$.

So,

$$a_2 = \frac{n_o K_o \sin\theta_o}{n_1 R_2} + \frac{d_1 \sin\theta_1}{R_2}. \tag{48}$$

Now, since $K_o \sin\theta_o = a_1 R_1$, we can finally write

$$a_2 = \frac{1}{R_2}\left[\frac{n_o}{n_1} a_1 R_1 + d_1 \sin\theta_1\right]. \tag{49}$$

If we need to calculate the distance K_2 of the final image formed by the second surface, we just substitute in the equation

$$K_2 = \frac{r_4}{\sin(-\theta_2)}. \tag{50}$$

This can be rewritten as

$$\frac{1}{K_2} = -\left(\frac{n_2}{n_1}\right)\frac{1}{a_2 R_2} \sin\theta_2. \tag{51}$$

So, we finally use Eq. (51) to calculate the image distance K_2, by just substituting for θ_2 and a_2 from Eqs. (45) and (49). Equation (51) can be easily derived starting from Eq. (50) as follows:

$$\frac{1}{K_2} = -\frac{1}{r_4} \sin\theta_2, \tag{52}$$

$$r_4 = \frac{n_1}{n_2} r_3 = \frac{n_1}{n_2}(r_2 + d_1 \sin\theta_1) = \frac{n_1}{n_2}(a_2 R_2). \tag{53}$$

Substituting from Eq. (53) into Eq. (52), we get Eq. (51).

For an optical system with several surfaces, two equations having the forms of Eqs. (45) and (49) are used for each surface, and finally, the image position of the last surface is calculated.

Summary of the procedure:

For the mth surface of a centered system of spherical surfaces we have the equations

$$a_m = \frac{1}{R_m}\left[\frac{n_{m-2}}{n_{m-1}} a_{m-1} R_{m-1} + d_{m-1} \sin\theta_{m-1}\right], \tag{54}$$

$$\theta_m = \theta_{m-1} - \sin^{-1} a_m + \sin^{-1}\frac{n_{m-1}}{n_m} a_m, \tag{55}$$

$$\frac{1}{K_m} = -\left(\frac{n_m}{n_{m-1}}\right)\frac{1}{a_m R_m} \sin\theta_m. \tag{56}$$

Remarks:

(1) For each surface of the optical system, we just use the first two equations, (54) and (55). The third equation, (56), is only used for the last surface.

(2) In this procedure, we consider the distances d_m between the successive curvature centers instead of the distances t_m separating these surfaces. So, all the distances d_m for the optical system must be given before applying the procedure. The following equation can be used to calculate the distance d_m in terms of the separation t_m between any two successive surfaces

$$d_{m-1} = t_{m-1} - (R_{m-1} - R_m). \quad (57)$$

(3) For the first surface ($m = 1$), there is no need to use Eq. (54), since $a_1 = \frac{K_o \sin\theta_o}{R_1}$ is already known because K_o, θ_o, and R_1 are all given.

C. Numerical Examples

Assume that we have a system of four centered spherical surfaces that form a final image I_4 for a real object O located on the optical axis of the system at a distance K_o from the curvature center C_1 of the first surface. Now, we need to calculate the exact distance K_4 of the final image I_4 if we know the following:

$\theta_o = 17.309724$, $K_o = 22$ cm, $R_1 = 10$ cm,

$R_2 = -8$ cm, $R_3 = 12$ cm, $R_4 = -10$ cm,

$t_1 = 5$ cm, $t_2 = 5$ cm, $t_3 = 8$ cm, $n_o = 1$,

$n_1 = 1.2$, $n_2 = 1$, $n_3 = 1.5$, $n_4 = 1$.

To solve this problem, we first need to calculate the distances d_m between the curvature centers using Eq. (57), where we easily get $d_1 = -13$ cm, $d_2 = 25$ cm, $d_3 = -14$ cm, $d_4 = 3$ cm. Also, for the first surface, the constant a_1 is given by Eq. (41) as $a_1 = 0.654581$. Now, for each surface we just substitute in Eqs. (54) and (55) to get the following:

first surface, $a_1 = 0.654581$, $\theta_1 = 9.479589$;
second surface, $a_2 = -0.414224$, $\theta_2 = 4.143784$;
third surface, $a_3 = 0.481920$, $\theta_3 = -5.926743$;
fourth surface, $a_4 = -0.530095$, $\theta_4 = -26.583586$.

The final image distance K_4 is given by Eq. (56) as

$$K_4 = 17.768432 \text{ cm}.$$

This numerical example proves the simplicity of this ray-tracing technique, and that it is the shortest in geometrical optics. In the previous calculations, all values are restricted to only seven digits. The higher the number of digits, the more precise is the result.

D. Another Procedure

In this section, we get another procedure in which we calculate the image distance for each optical surface. The procedure consists of two equations for each surface of the optical system. In Eqs. (54)–(56), and for simplicity, consider the relative refractive indices: $\mu_1 = n_1/n_o$, $\mu_2 = n_2/n_1$, and so on. Now, from Eq. (56) we get

$$\frac{1}{K_{m-1}} = -\frac{\mu_{m-1}}{a_{m-1}R_{m-1}} \sin\theta_{m-1}. \quad (58)$$

Substituting for $\sin\theta_{m-1}$ into (54) we get

$$a_m = \frac{a_{m-1}R_{m-1}}{\mu_{m-1}R_m}\left[1 - \frac{d_{m-1}}{K_{m-1}}\right]. \quad (59)$$

Also, we substitute for θ_{m-1} from Eqs. (58) into (55) to get

$$\theta_m = \sin^{-1}\frac{a_m}{\mu_m} - \sin^{-1} a_m - \sin^{-1}\frac{a_{m-1}R_{m-1}}{\mu_{m-1}K_{m-1}}. \quad (60)$$

Substituting from Eqs. (60) into (56) we get

$$\frac{1}{K_m} = \frac{\mu_m}{a_m R_m} \sin\left[\sin^{-1} a_m - \sin^{-1}\frac{a_m}{\mu_m} + \sin^{-1}\frac{a_{m-1}R_{m-1}}{\mu_{m-1}K_{m-1}}\right]. \quad (61)$$

Therefore, we have a new ray-tracing procedure in which we use just the two equations, (59) and (61).

E. Paraxial Formula for a System of Spherical Surfaces

In the previous procedure, we noted that the values of a_{m-1}, a_{m-2}, a_{m-3}, etc., can be obtained by Eq. (59). Now let us assume that

$$D_m = \left[1 - \frac{d_m}{K_m}\right]. \quad (62)$$

So, Eq. (59) can have the form

$$a_m = \frac{n_o a_1 R_1}{n_{m-1}R_m} D_{m-1}D_{m-2}D_{m-3}\ldots.D_1. \quad (63)$$

Now, in the paraxial region, $\theta_o \simeq 0$ and $a_1 \simeq 0$ in Eq. (41). Thus using Eq. (63), we find that $a_m \simeq 0$ for all values of m. Therefore, in the paraxial region, Eq. (61) can be written as

$$\frac{1}{K_m} = \frac{\mu_m}{a_m R_m}\left[a_m - \frac{a_m}{\mu_m} + \frac{a_{m-1}R_{m-1}}{\mu_{m-1}K_{m-1}}\right]. \quad (64)$$

Combining Eqs. (59) and (62), we get

$$a_m = \frac{a_{m-1}R_{m-1}}{\mu_{m-1}R_m} D_{m-1}. \quad (65)$$

Substituting for a_{m-1} from Eqs. (65) into (64) we get

$$\frac{1}{K_m} = \frac{\mu_m}{R_m}\left[1 - \frac{1}{\mu_m} + \frac{R_m}{D_{m-1}K_{m-1}}\right]. \quad (66)$$

From Eq. (62) we get $D_{m-1}K_{m-1} = [K_{m-1} - d_{m-1}]$, and Eq. (66) can be rewritten as

$$\frac{1}{K_m} + \frac{\mu_m}{d_{m-1} - K_{m-1}} = \frac{\mu_m - 1}{R_m}. \quad (67)$$

Since $\mu_m = n_m/n_{m-1}$, Eq. (67) can have the final form

$$\frac{n_{m-1}}{K_m} + \frac{n_m}{d_{m-1} - K_{m-1}} = \frac{n_m - n_{m-1}}{R_m}. \quad (68)$$

Equation (68) is the paraxial formula that can be used to trace a paraxial ray through a system of spherical surfaces and to calculate the position of the final image formed by the system. This will be shown in a numerical example. It should be noted that for the first optical surface ($m = 1$), the distance K_o in Eq. (68) must be negative, since it represents the distance of a virtual image formed by an imaginary surface ($m = 0$) preceding the first surface. Since the image is located to the left of the curvature center, K_o is negative according to our sign convention. Thus, for the first surface, Eq. (68) can be written as

$$\frac{n_o}{K_1} + \frac{n_1}{K_o} = \frac{n_1 - n_o}{R_1}. \quad (69)$$

F. Numerical Example

Consider a centered system of four spherical surfaces. Let us calculate the final paraxial image I_4 formed by the system for a point object O placed at a distance K_o from the curvature

center of the first surface. The following are given: $K_o = 22$ cm, $R_1 = 10$ cm, $R_2 = -8$ cm, $R_3 = 12$ cm, $R_4 = -10$ cm, $t_1 = 5$ cm, $t_2 = 5$ cm, $t_3 = 8$ cm. $n_o = 1$, $n_1 = 1.2$, $n_2 = 1$, $n_3 = 1.5$, $n_4 = 1$. To solve this problem, we also need to calculate the distances d_m between the curvature centers using Eq. (57), where we get $d_1 = -13$ cm, $d_2 = 25$ cm, and $d_3 = -14$ cm. Now we apply Eq. (68) for each surface to get the position of the image formed by this surface. So, we easily get the following:

first surface, $K_1 = -28.947368$ cm;
second surface, $K_2 = -31.825173$ cm;
third surface, $K_3 = 65.487885$ cm;
fourth surface, $K_4 = 40.087735$ cm.

(1) This method is different from that used in Gaussian optics. For an optical system of n surfaces, the final image position relative to the vertex of the last surface is given by

$$S_m = K_m - R_m.$$

(2) We can easily calculate the exact value for the longitudinal spherical aberration of a system of n spherical surfaces using the previous procedure, i.e., Eq. (68), and the exact procedure, Eqs. (54)–(56).

7. THICK LENS MERIDIONAL EQUATION

A. Description

In Fig. 5, a point object O is placed on the optical axis of a thick lens at a distance K_o from the curvature center C_1 of the first surface. The final image I formed by the lens is located at a distance K_2 from the curvature center C_2 of the second surface. The meridional ray OP makes an angle θ_o with the optical axis.

In this section, we prove that the exact meridional equation for this lens has the form

$$\frac{1}{K_2} = \frac{n_2}{n_o B} \sin\left[\sin^{-1}\frac{n_o B}{n_1 R_2} - \sin^{-1}\frac{n_o B}{n_2 R_2} - \sin^{-1}\frac{n_o}{n_1}\left(\frac{B - a_1 R_1}{d_1}\right)\right], \tag{70}$$

where

$$B = \left[a_1 R_1 + \frac{n_1 d_1}{n_o} \sin\left(\theta_o - \sin^{-1} a_1 + \sin^{-1}\frac{n_o}{n_1} a_1\right)\right] \tag{71}$$

and

$$a_1 = \frac{K_o \sin\theta_o}{R_1}. \tag{72}$$

Equation (70) can be reduced to the Gaussian (paraxial) thick lens equation, as will be proved in this section. To derive Eq. (70), we apply Eqs. (54) and (55) as follows: for the first surface ($m = 1$) we have

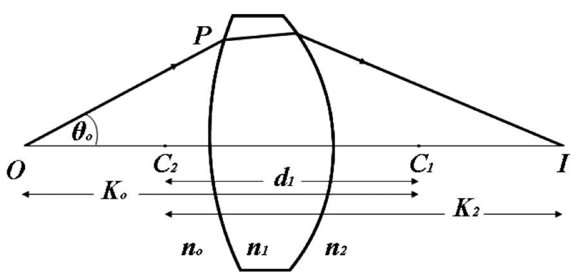

Fig. 5. Propagation of a meridional ray in a thick lens.

$$a_1 = \frac{K_o \sin\theta_o}{R_1}, \tag{73}$$

$$\theta_1 = \theta_o - \sin^{-1} a_1 + \sin^{-1}\frac{n_o}{n_1} a_1. \tag{74}$$

For the second surface ($m = 2$) we have

$$a_2 = \frac{1}{R_2}\left(\frac{n_o}{n_1} a_1 R_1 + d_1 \sin\theta_1\right), \tag{75}$$

$$\theta_2 = \theta_1 - \sin^{-1} a_2 + \sin^{-1}\frac{n_1}{n_2} a_2. \tag{76}$$

The image location K_2 is given by Eq. (56) as

$$\frac{1}{K_2} = -\left(\frac{n_2}{n_1}\right)\frac{1}{a_2 R_2} \sin\theta_2. \tag{77}$$

Substituting from Eq. (76) into Eq. (77) we get

$$\frac{1}{K_2} = \frac{-n_2}{n_1 a_2 R_2} \sin\left[\theta_1 - \sin^{-1} a_2 + \sin^{-1}\frac{n_1}{n_2} a_2\right]. \tag{78}$$

From Eqs. (74) and (75) we have

$$a_2 = \frac{n_o}{n_1 R_2}\left[a_1 R_1 + \frac{n_1 d_1}{n_o} \sin\left(\theta_o - \sin^{-1} a_1 + \sin^{-1}\frac{n_o}{n_1} a_1\right)\right]. \tag{79}$$

Assuming that

$$B = \left[a_1 R_1 + \frac{n_1 d_1}{n_o} \sin\left(\theta_o - \sin^{-1} a_1 + \sin^{-1}\frac{n_o}{n_1} a_1\right)\right], \tag{80}$$

Eq. (79) becomes

$$a_2 = \frac{n_o}{n_1 R_2} B. \tag{81}$$

Also from Eqs. (74) and (80) we have

$$\theta_1 = \sin^{-1}\left[\frac{n_o}{n_1}\left(\frac{B - a_1 R_1}{d_1}\right)\right], \tag{82}$$

Substituting from Eqs. (81) and (82) into Eq. (78), we get Eq. (70).

B. Numerical Example

For the thick lens of Fig. 5, we have the following data: $R_1 = 10$ cm, $R_2 = -8$ cm, $\theta_o = 17.309724°$, $K_o = 22$ cm, $n_o = 1$, $n_1 = 1.2$, $n_2 = 1$ and $d = -13$ cm. To calculate K_2, we substitute into Eqs. (70)–(72), and we get $a_1 = 0.654581$, $B = 3.976547$, and $K_2 = -55.031208$ cm.

C. Parallel Incidence

If the object O is at infinity (i.e., $K_o = \infty$ and $\theta_o = 0$), Eq. (72) reduces to $a_1 = h/R_1$ and the lens is described by the following equation:

$$\frac{1}{K_2} = \frac{n_2}{n_o B} \sin\left[\sin^{-1}\frac{n_o B}{n_1 R_2} - \sin^{-1}\frac{n_o B}{n_2 R_2} - \sin^{-1}\frac{n_o}{n_1}\left(\frac{B - h}{d_1}\right)\right], \tag{83}$$

where

$$B = \left[h + \frac{n_1 d_1}{n_o} \sin\left(\sin^{-1}\frac{n_o h}{n_1 R_1} - \sin^{-1}\frac{h}{R_1}\right)\right]. \tag{84}$$

N.B.: If the lens is placed in air (i.e., $n_o = 1$, $n_1 = n$ and $n_2 = 1$), Eqs. (83) and (84) will have the simpler forms

$$\frac{1}{K_2} = \frac{1}{B}\sin\left[\sin^{-1}\frac{B}{nR_2} - \sin^{-1}\frac{B}{R_2} - \sin^{-1}\frac{B-h}{nd_1}\right], \tag{85}$$

where

$$B = \left[h + nd_1 \sin\left(\sin^{-1}\frac{h}{nR_1} - \sin^{-1}\frac{h}{R_1}\right)\right]. \tag{86}$$

So, Eq. (85) is an exact meridional thick (or thin) lens equation in case of parallel incidence.

D. Paraxial Approximation

In the paraxial region, the exact lens formula, Eq. (70), reduces to the following paraxial form:

$$\frac{1}{K_2} + \frac{n_2}{n_o K_o E} = \left(\frac{n_2}{n_o} - \frac{n_2}{n_1}\right)\frac{1}{R_1 E} - \left(\frac{n_1 - n_2}{n_1 R_2}\right), \tag{87}$$

where

$$E = \left[1 - \left(\frac{n_1}{n_o} - 1\right)\frac{d_1}{R_1} + \frac{n_1 d_1}{n_o K_o}\right]. \tag{88}$$

This can be proved by making use of the fact that in the paraxial region, $\theta_o \simeq 0$. So, from Eqs. (71) and (72), we get $a_1 \simeq 0$ and $B \simeq 0$. Therefore, Eq. (70) reduces to

$$\frac{1}{K_2} = \frac{n_2}{n_o B}\left[\frac{n_o B}{n_1 R_2} - \frac{n_o B}{n_2 R_2} - \frac{n_o}{n_1}\left(\frac{B - a_1 R_1}{d_1}\right)\right]. \tag{89}$$

This can be rewritten as

$$\frac{1}{K_2} = \frac{1}{R_2}\left(\frac{n_2}{n_1} - 1\right) - \left(\frac{n_2}{n_1 d_1}\right)\left[1 - \frac{a_1 R_1}{B}\right]. \tag{90}$$

From Eq. (72), we get $\theta_o \simeq a_1 R_1 / K_o$. So, Eq. (71) reduces to

$$B = a_1 R_1 \left[1 - \left(\frac{n_1}{n_o} - 1\right)\frac{d_1}{R_1} + \frac{n_1 d_1}{n_o K_o}\right]. \tag{91}$$

Now, let us assume that

$$E = \left[1 - \left(\frac{n_1}{n_o} - 1\right)\frac{d_1}{R_1} + \frac{n_1 d_1}{n_o K_o}\right]. \tag{92}$$

Then, Eq. (91) can be written as

$$B = a_1 R_1 E. \tag{93}$$

Substituting into Eq. (90), we get

$$\frac{n_1 E}{K_2} = \left(\frac{n_2 - n_1}{R_2}\right)E - \left(\frac{n_2}{d_1}\right)[E - 1]. \tag{94}$$

Substituting from Eq. (92) into Eq. (94) and rearranging terms, we finally get Eqs. (87) and (88).

N.B.: If the lens is surrounded by air (i.e., $n_o = n_2 = 1$ and $n_1 = n$), then Eq. (87) becomes

$$\frac{1}{K_2} + \frac{1}{K_o E} = \left(\frac{n-1}{n}\right)\left[\frac{1}{R_1 E} - \frac{1}{R_2}\right], \tag{95}$$

where

$$E = \left[1 - (n-1)\frac{d_1}{R_1} + \frac{nd_1}{K_o}\right]. \tag{96}$$

E. Power of a Thick Lens

In the literature, the power P of a lens is simply the inverse of the focal length. Now, if we let $K_o \simeq \infty$ in Eqs. (87) and (88), the distance K_2 represents the focal length of the lens measured from the rear focus to the curvature center of the second surface. So, it can be replaced by the symbol f_c. Thus, we have

$$\frac{1}{f_c} = \left(\frac{1}{n_o} - \frac{1}{n_1}\right)\frac{n_2}{R_1 E} - \left(\frac{n_1 - n_2}{n_1 R_2}\right), \tag{97}$$

where

$$E = \left[1 - \left(\frac{n_1}{n_o} - 1\right)\frac{d_1}{R_1}\right]. \tag{98}$$

If the lens is surrounded by air (i.e., $n_o = n_2 = 1$ and $n_1 = n$), then Eqs. (97) and (98) take the simple form

$$\frac{1}{f_c} = \left(\frac{n-1}{n}\right)\left(\frac{1}{R_1 E_o} - \frac{1}{R_2}\right), \tag{99}$$

where

$$E_o = \left[1 - \frac{(n-1)d_1}{R_1}\right]. \tag{100}$$

So, combining Eqs. (99) and (100), we get

$$\frac{1}{f_c} = \left(\frac{n-1}{n}\right)\left(\frac{1}{R_1 - (n-1)d_1} - \frac{1}{R_2}\right). \tag{101}$$

Equation (101) is a new form for the paraxial thick lens equation. In this equation, the quantity $\frac{1}{f_c}$ can be considered as the power of the lens. In this case, we have to define a new power unit. In the Gaussian formula, the power is $1/f_p$, where f_p is the focal length measured to the principal plane. In a previous work [7], we proved that the relation between f_c and f_p is

$$f_c = E_o f_p. \tag{102}$$

N.B.: Eq. (101) can be derived from Eq. (68) in much simpler steps.

8. LONGITUDINAL SPHERICAL ABERRATION

A. For a Thick (Thin) Len

The longitudinal spherical aberration (Long. SA), of a thin lens is measured by the difference

$$(\text{Long. SA}) = S_p - S_n. \tag{103}$$

In Eq. (103), S_p is the distance of the image formed by the paraxial ray measured to the lens, while S_n is the corresponding distance of the image formed by a marginal ray incident at a height h. Equation (103) can be written as

$$(\text{Long. SA}) = S_p S_n \left(\frac{1}{S_n} - \frac{1}{S_p}\right) = S_p S_n L_s. \tag{104}$$

In fact, the Long. SA can be minimized if the quantity L_s is minimized. So, it is useful to derive an exact formula for the quantity L_s of a thin or thick lens. In the present work, distances are measured to the center of curvature, so Eq. (104) will have the form

$$\begin{aligned}(\text{Long. SA}) &= (K_2)_p (K_2)_n \left(\frac{1}{(K_2)_n} - \frac{1}{(K_2)_p}\right)\\ &= (K_2)_p (K_2)_n L_s,\end{aligned} \tag{105}$$

i.e.,

$$L_s = \frac{1}{(K_2)_n} - \frac{1}{(K_2)_p}. \tag{106}$$

$(K_2)_n$ and $(K_2)_p$ are the distances corresponding to S_n and S_p, respectively. Now we write Eqs. (85) and (101) as

$$\frac{1}{(K_2)_n} = \frac{1}{B}\sin\left[\sin^{-1}\frac{B}{nR_2} - \sin^{-1}\frac{B}{R_2} - \sin^{-1}\frac{B-h}{nd_1}\right], \tag{107}$$

$$\frac{1}{(K_2)_p} = \left(\frac{n-1}{n}\right)\left(\frac{1}{R_1-(n-1)d_1} - \frac{1}{R_2}\right). \tag{108}$$

Substituting from Eqs. (107) and (108) into Eq. (106), we get

$$\left.\begin{array}{l} L_s = \frac{1}{B}\sin\left[\sin^{-1}\frac{B}{nR_2} - \sin^{-1}\frac{B}{R_2} - \sin^{-1}\frac{B-h}{nd_1}\right] \\ -\left(\frac{n-1}{n}\right)\left(\frac{1}{R_1-(n-1)d_1} - \frac{1}{R_2}\right) \end{array}\right\}, \tag{109}$$

where

$$B = \left[h + nd_1 \sin\left(\sin^{-1}\frac{h}{nR_1} - \sin^{-1}\frac{h}{R_1}\right)\right]. \tag{110}$$

To the best of our knowledge, Eq. (109) is new, and it may have a possible theoretical importance.

Remarks:

(1) All the equations concerning a thick lens, including Eq. (109), can be applied for the case of a thin lens by just neglecting the thickness t in the equation $d_1 = t - (R_1 - R_2)$.

(2) In the previous equations, the lens is assumed to be surrounded by air, and n is its refractive index. For the case of three refractive indices n_o, n_1, and n_2, as in Fig. 5, Eq. (109) can be easily generalized using Eqs. (83), (84), (97), and (98).

B. For a Refracting Spherical Surface

An exact formula for L_s of a refracting spherical surface can be obtained using Eqs. (14) and (18) and also Eqs. (14) and (19) as follows:

$$L_s = \frac{n}{h}\sin\left[\sin^{-1}\frac{h}{R} - \sin^{-1}\frac{h}{nR}\right] - \frac{n-1}{R}, \tag{111}$$

$$L_s = \pm\frac{1}{R^2}\left[\sqrt{n^2R^2-h^2} - \sqrt{R^2-h^2}\right] - \frac{n-1}{R}. \tag{112}$$

Equations (111) and (112) are two different exact formulas for L_s of a refracting spherical surface.

C. For a Spherical Mirror

Using Eqs. (24) and (35), we can easily get an exact formula for L_s of a spherical mirror as follows:

$$L_s = \frac{2}{R}\sqrt{1-\left(\frac{h}{R}\right)^2} - \frac{2}{R}. \tag{113}$$

9. CONCLUSION

In this work, our target was to derive general ray-tracing formulas that can be reduced to simple paraxial equations. This could be easily achieved because the positions of the object and the images were measured relative to the curvature centers of the surfaces, not to the principal planes. The obtained formulas are new and can be easily used by optics students as generalized forms of the Gaussian equations. The work is just an approach toward a much more perfect theory for nonparaxial optics that may appear in the future.

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